Threshold Strategy for a Leaking Corner-Free Hamilton-Jacobi Reachability with Decomposed Computations

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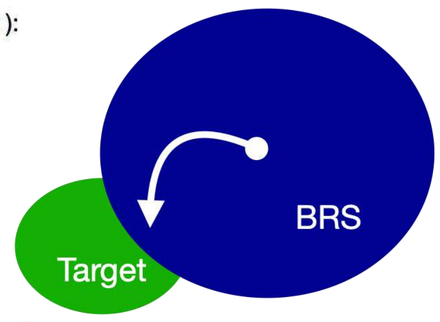
BACKGROUND: HAMILTON-JACOBI REACHABILITY

- Implicit Surface Function (0 sublevel set is the target set):

$$z \in \mathcal{T} \leftrightarrow \ell(z) \leq 0$$



- System Dynamics: $\dot{z} = f(z) + g(z) \cdot u$
- Trajectory: $\ell(\zeta(0; z, t, u(\cdot))$
- Value Functions:

Liveness problem: $V(z,t) = \min_{u(\cdot)\in\mathbb{U}} \ell(\zeta(0; z, t, u(\cdot)))$

Safety problem: $V(z,t) = \max_{u(\cdot)\in\mathbb{U}} \ell(\zeta(0; z, t, u(\cdot)))$

- Backward computation with HJ-PDE (Grid-based Dynamic Programming):

Liveness problem: $V(z, t-\delta t) = V(z,t) + \min_{\dot{z}} D_z V(z,t)\dot{z}\delta t,$

Safety problem: $V(z, t-\delta t) = V(z,t) + \max_{\dot{z}} D_z V(z,t)\dot{z}\delta t, \quad V(z,0) = l(z)$

- Backward Reachable Set (BRS):

$$z \in \mathcal{R}(t) \leftrightarrow V(z,t) \leq 0$$

- Computationally expensive due to the curse of dimensionality

PROBLEM: LEAKING CORNER ISSUE

Definition: (Leaking Corners) Suppose we obtain $V(z,t)$ by solving HJ-PDE, and $\hat{V}(z,t)$ by combining value functions. The set of "leaking corners" $\mathcal{L}(t)$ is defined as

$$\mathcal{L}(t) = \{z : V(z,t) \neq \hat{V}(z,t)\}$$

2 Cases will suffer from the issue:

(1) Intersection case for liveness problem: $\hat{V}_R(z,t) = \max\{V_{R,1}(x_1,t), V_{R,2}(x_2,t)\}$

(2) Union case for safety problem: $\hat{V}_A(z,t) = \min\{V_{A,1}(x_1,t), V_{A,2}(x_2,t)\}$

METHOD: THRESHOLD STRATEGY

Theorem 1: We can find the set of leaking corners $\mathcal{L}(t)$ by comparing the (full-dimensional) sub-value functions.

$$\mathcal{L}(t) = \{z : |V_1 - V_2| < \Delta\}.$$

The value of Δ is

$$\Delta = \begin{cases} |\tilde{V}_1^* - V_1|, & \text{if } V_{R,2} \geq V_{R,1} \text{ or } V_{A,1} \geq V_{A,2} \\ |\tilde{V}_2^* - V_2|, & \text{if } V_{R,1} \geq V_{R,2} \text{ or } V_{A,2} \geq V_{A,1}. \end{cases}$$

EXAMPLE METHOD SUFFERS FROM THE LEAKING CORNER ISSUE

Self-contained Subsystem Decomposition
- Computation happens in low dimensions
- Solution to the Curse of Dimensionality

- Full-dimensional system: 2D Single Integrator

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- Control Input:

$u = (u_x, u_y)$; constrained by $c(u) = \|u\|_2 - \bar{u} \leq 0$

- Value Function: $V(z,t)$

- Subsystem 1:

$$\dot{x}_1 = \dot{x} = u_x$$

- Control Input:

$w_x = u_x$; constrained by $c_1(w_x) = \|u_x\|_2 - \bar{u} \leq 0$

- Value Function: $V_1(z,t) = \phi_1(x_1,t)$

- Subsystem 2:

$$\dot{x}_2 = \dot{y} = u_y$$

- Control Input:

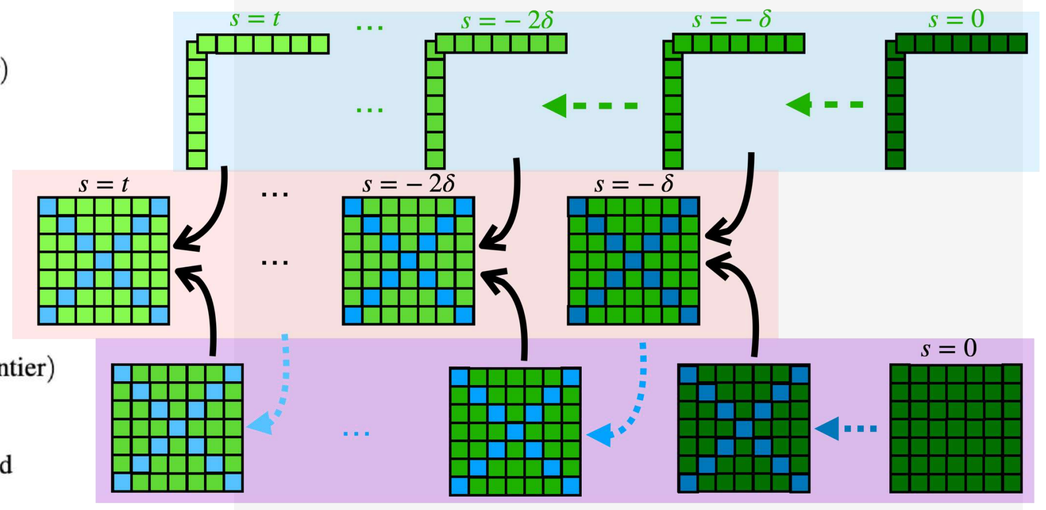
$w_y = u_y$; constrained by $c_2(w_y) = \|u_y\|_2 - \bar{u} \leq 0$

- Value Function: $V_2(z,t) = \phi_2(x_2,t)$

Algorithm 1: Local updating procedure

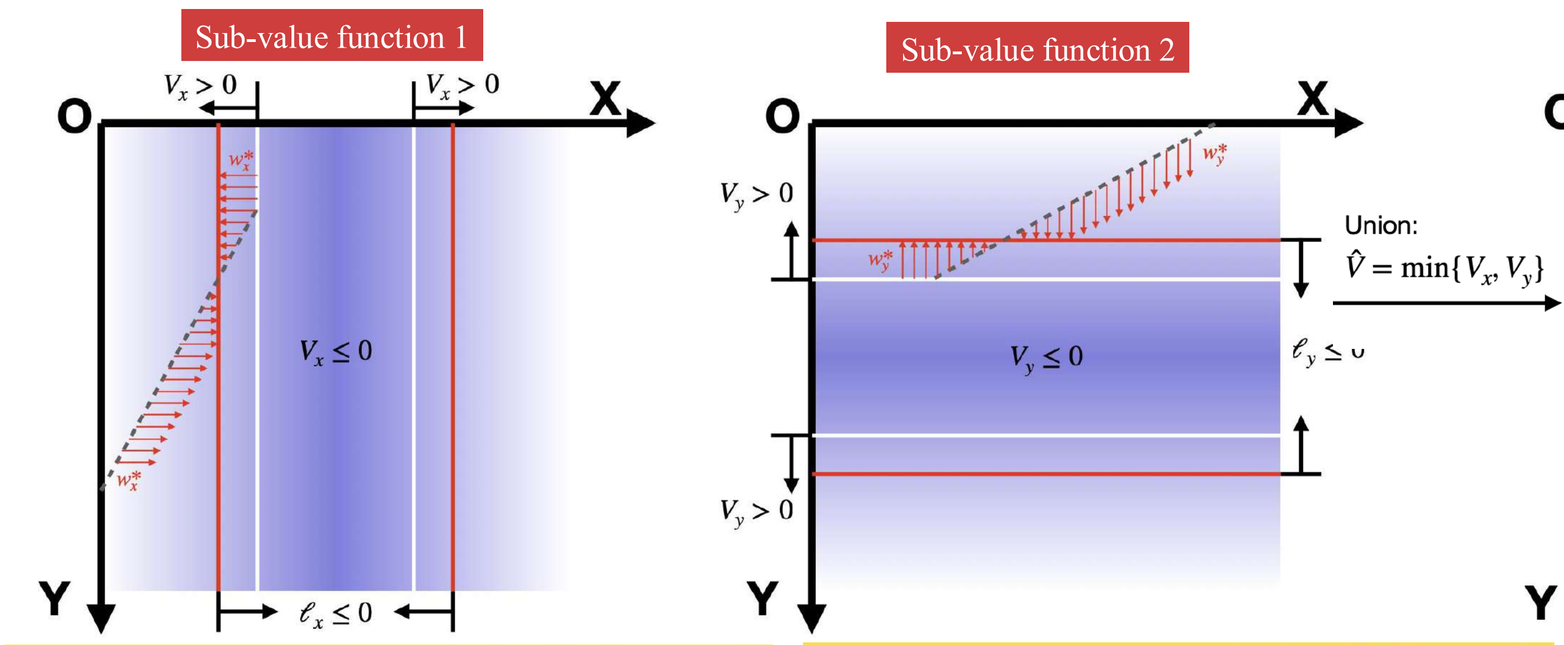
Data: $\hat{V}(\cdot,\cdot), \hat{\mathcal{L}}(\cdot), Z, t_{\text{list}} = [t, t+\delta..., 0]$
Result: $\check{V}(\cdot,\cdot)$
$s \leftarrow 0$; ▷ Backward Computation
$\check{V}(\cdot,0) \leftarrow \hat{V}(\cdot,0)$;
Frontier ← nextFrontier ← visited ← {};
while $s > t$ **do**
  for $z \in Z$ **do**
    if $z \in \hat{\mathcal{L}}(s)$ **then**
      updateValue$(z, s, \delta,$ Frontier$)$
    else
      $\check{V}(z, s-\delta) \leftarrow \hat{V}(z, s-\delta)$;
    end
  end
  visited ← $\hat{\mathcal{L}}(s)$
  Frontier ← Frontier \ visited
  while *Frontier* $\neq \emptyset$ **do**
    for z *in Frontier* **do**
      updateValue$(z, s, \delta,$ nextFrontier$)$
    end
    visited ← visited ∪ Frontier
    Frontier ← nextFrontier \ visited
    nextFrontier ← {}
  end
  $s \leftarrow s - \delta$;
end
def *updateValue*$(z, s, \delta,$ *Frontier*$)$**:**
  $\check{V}(z, s-\delta) \leftarrow$ HJ Update$(\check{V}(z,s))$ ▷ Equation 5
  if $\check{V}(z, s-\delta) \neq \hat{V}(z, s-\delta)$ **then**
    Frontier ← Frontier ∪ neighbor(z)
  end

METHOD: LOCAL UPDATING PROCEDURE



CONTROL INCONSISTENCY IN UNION SET- CAUSE OF THE LEAKING CORNER ISSUE

Avoiding zero-sublevel set



When w_x and w_y both acting towards optimal, the full-dimensional control constraint will be violated.

$u = (u_x, u_y)$;

Full-dimensional control constraint:

$c(u) = \|u\|_2 - \bar{u} \leq 0$

$w_x = u_x$; constrained by $c_1(w_x) = \|u_x\|_2 - \bar{u} \leq 0$

$w_y = u_y$; constrained by $c_2(w_y) = \|u_y\|_2 - \bar{u} \leq 0$

RESULT (2D SINGLE INTEGRATOR)

TABLE I: 2D Accuracy Comparison for One Step

Metric	Before	After
Number of grid points with different values from the ground truth	200	0
Average absolute difference from ground truth	1.2×10^{-4}	9.51×10^{-18}
Maximum absolute difference from ground truth	2×10^{-2}	2.22×10^{-16}

TABLE III: 2D Accuracy Comparison for 10 Steps

Metric	Before	After
Number of states with different values from the ground truth	1344	0
Average absolute difference from ground truth	2.5×10^{-3}	1×10^{-9}
Maximum absolute difference from ground truth	7.39×10^{-2}	2.44×10^{-8}

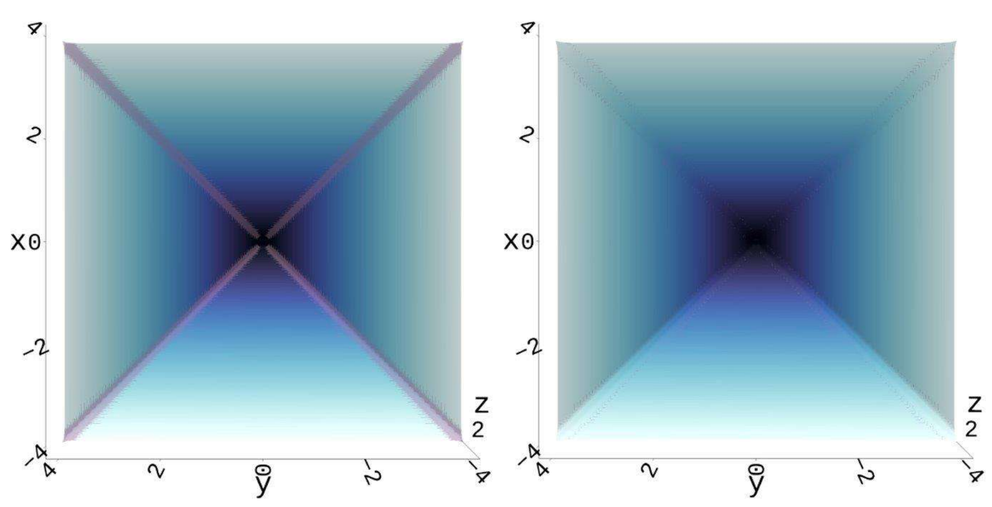


TABLE II: 2D Time Comparison for One Step

Process	Time (seconds)
Direct computation	3.3×10^{-2}
SCSD computation + HJ local update computation	$7 \times 10^{-4} + 1.3 \times 10^{-3} = 2.0 \times 10^{-3}$

TABLE IV: 2D Time Comparison for 10 Steps

Process	Time (seconds)
Direct computation	3.36×10^{-1}
SCSD computation + HJ local update computation	$4.72 \times 10^{-3} + 1.61 \times 10^{-1} = 1.65 \times 10^{-1}$

RESULT (6D PLANAR QUADROTOR)

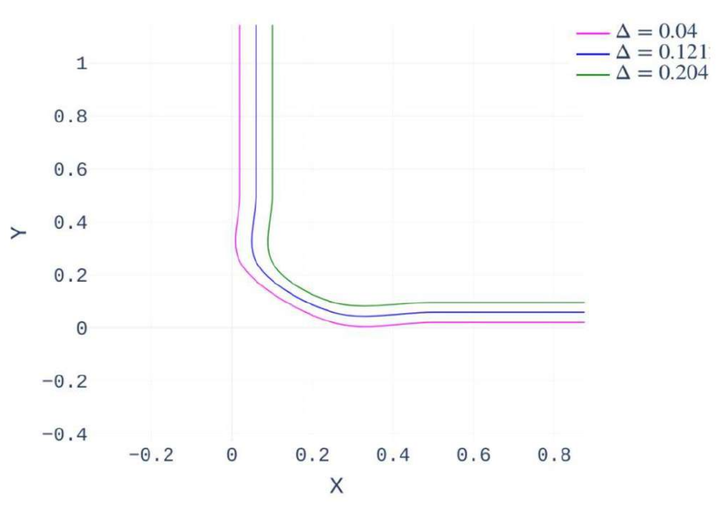


TABLE VI: Computation Time and Delta Value for ts

t (s)	Δ	Decomposition Time + Local Updating Time (seconds)
-0.02	0.04	2.447 + 47.1078 = 49.5548
-0.06	0.1212	6.769 + 157.3528 = 164.1218
-0.1	0.204	11.8038 + 250.7859 = 262.5897

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For math proofs, check the paper with QR code.
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